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LONG WAVES IN SHALLOW LIQUID UNDER ICE COVER*

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Long waves in shallow liquid under ice cover which is under tension or compression are studied. An equation describing the propagation of such waves is obtained first. Exact solutions of the equation in the form of cnoidal waves and solitons are derived and studied. It is shown that in the case of sufficiently low tension and in all cases of compression the periodic waves in a shallow liquid are unstable and collapse.

Waves in a liquid of finite depth under ice cover have been studied by many workers in the linear approximation (see e.g. /1, 2/). The influence of non-linearity on the propagation of periodic waves and wave packets of finite intensity was first studied in /3, 4/, where it was shown that periodic waves are unstable in a deep fluid and collapse under any compressive-tensile loads acting on the ice cover that may be encountered in practice.

1. Let us consider the motion of a heavy liquid under ice cover, which we shall model by a thin elastic plate. The behaviour of the plate under the action of external loads is described by the equation /5/

$$\frac{Eh^3}{12(1-\nu^2)}\eta_{xxxx} - h\sigma_{xx}\eta_{xx} = P - \rho_i h\eta_{tt}$$

where η is the deviation of the middle plane of the plate from its equilibrium position, P denotes the external load, and ρ_i and h are the density and thickness of the ice. We will assume that the ice cover can be under compression-tension along the horizontal x axis, characterized by the component of the stress tensor $\sigma_{xx} = \text{const}$.

The equations of motion of the liquid with boundary conditions at the bottom and under the elastic plate are /3, 4/:

$$-H < z < \eta,\ \varphi_{xx} + \varphi_{zz} = 0;\ z = -H,\ \varphi_z = 0 \tag{1.1}$$

$$z = \eta,\quad \eta_t + \varphi_x\eta_x = \varphi_z,\quad \varphi_t + {}^1/_2(\varphi_x^2 + \varphi_z^2) + g\eta + M\eta_{xxxx} - K\eta_{xx} + L\eta_{tt} = 0$$

Here φ is the velocity potential of the liquid and H is its depth.

From (1.1) we obtain, in the linear approximation, the following dispersion equation connecting the frequency ω and the wavelength $2\pi\lambda$ (ρ_w is the density of water)

$$\omega^2(L + (k\,\text{th}\,kH)^{-1}) = g + Mk^4 + Kk^2,\ k \equiv \lambda^{-1} \tag{1.2}$$

$$M = \frac{Eh^3}{12(1-\nu^2)\rho_w},\quad L = \frac{\rho_i}{\rho_w}h,\quad K = \frac{h\sigma_{xx}}{\rho_w}$$

We shall consider the motions for which

$$\lambda \approx 100\ \text{m},\ H \approx 10\ \text{m},\ h \simeq 3\ \text{m},\ E \approx 3\cdot 10^9\ \text{Nm}^{-2}$$
$$\sigma_{xx} = 10^4 - 10^5\ \text{Nm}^{-2}$$

Within such scales the quantities

$$\gamma \equiv \frac{M}{g\lambda^4},\quad \beta \equiv \frac{K}{g\lambda^2},\quad \delta \equiv \frac{LH}{\lambda^2},\quad \mu \equiv \frac{H^2}{\lambda^2}$$

are of the order of 10^{-3}, therefore we can write the dispersion relation (1.2) with an accuracy of up to $O(10^{-3})$ in the form

$$\omega = k\sqrt{H}\left(\sqrt{g} - \sqrt{g}\frac{k^2H^2}{6} - \sqrt{g}\frac{LHk^2}{2} + \frac{Mk^4}{2\sqrt{g}} + \frac{Kk^2}{2\sqrt{g}}\right) \tag{1.3}$$

The equation from which the dispersion law (1.3) follows, must have the following form:

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$$\eta_t + \sqrt{gH}\,\eta_x + \frac{\sqrt{H}}{2}\left(\sqrt{g}\,\frac{H^2}{3} + \sqrt{g}\,LH - \frac{K}{\sqrt{g}}\right)\eta_x^{(3)} + \frac{M\sqrt{H}}{2\sqrt{g}}\,\eta_x^{(5)} = 0 \tag{1.4}$$

It is interesting to take into account the non-linear corrections to Eq.(1.4) which could substantially affect the solution after a fairly long interval and at large distances. In order to obtain a non-linear analogue of (1.4), we shall introduce a small parameter ε, characterizing the non-linear form of the problem: $\varepsilon = aH^{-1}$, where a is the characteristic amplitude of the oscillations of the ice cover. Thus we find that in the non-linear problem in question five small dimensionless parameters $\varepsilon, \beta, \gamma, \delta, \mu$, appear, which may have different values depending on the characteristic scales, and may influence the properties of the propagating waves to a different extent. We shall assume that ε^2 is much smaller then the remaining parameters, i.e. that the characteristic amplitude $a < 0{,}1$ m.

Let us change in (1.1) to dimensionless quantities

$$x' = \frac{x}{\lambda},\quad z' = \frac{z}{H},\quad t' = \frac{\sqrt{gH}}{\lambda}t,\quad \varphi' = \frac{\sqrt{gH}}{ga\lambda}\varphi,\quad \eta' = \frac{\eta}{a}$$

As a result we obtain the following system of equations with boundary conditions (in what follows, the primes will be omitted):

$$\begin{aligned} &-1 < z < \varepsilon\eta,\ \mu\varphi_{xx} + \varphi_{zz} = 0;\ z = -1,\ \varphi_z = 0 \\ &z = \varepsilon\eta,\ \eta_t + \varepsilon\eta_x\varphi_x = \mu^{-1}\varphi_z \\ &\varphi_t + \eta + {}^1/_2\varepsilon(\varphi_x^2 + \mu^{-1}\varphi_z^2) + \gamma\eta_{xxxx} - \beta\eta_{xx} + \delta\eta_{tt} = 0 \end{aligned} \tag{1.5}$$

Expanding $\varphi(x, t, z)$ in a Taylor series in powers of z we obtain, from the Laplace equation and boundary condition when $z = -1$, the following expression, using the method of successive approximations:

$$\varphi_z^\circ = -\mu\varphi_{xx}^\circ - {}^1/_3\mu^2\varphi_{xxxx}^\circ \tag{1.6}$$

The superscript ($^\circ$) means that the value of the function or its derivatives is taken at $z = 0$.

Substitution of (1.6) into the boundary conditions with $z = \varepsilon\eta$ yields the following system of equations with an accuracy up to terms of order $O(\varepsilon, \mu)$:

$$\begin{aligned} &\varphi_t^\circ + \eta + {}^1/_2\varepsilon\varphi_x^{\circ 2} + \gamma\eta_{xxxx} - \beta\eta_{xx} + \delta\eta_{tt} = 0 \\ &\eta_t + \varepsilon\eta_x\varphi_x^\circ + \varphi_{xx}^\circ + {}^1/_3\mu\varphi_{xxxx}^\circ + \varepsilon\eta\varphi_{xx}^\circ = 0 \end{aligned} \tag{1.7}$$

In the zeroth approximation in $\varepsilon, \mu, \gamma, \beta, \delta$ (1.7) yields

$$\varphi_t^\circ + \eta = 0,\quad \eta_t + \varphi_{xx}^\circ = 0 \tag{1.8}$$

Henceforth, we shall only consider the wave moving to the right, i.e. we shall assume that $\partial/\partial t = -\partial/\partial x + O(\varepsilon, \mu)$. Then we can eliminate φ° from (1.7) and obtain the following equation in η:

$$\begin{aligned} &\eta_t + \eta_x + {}^3/_2\varepsilon\eta\eta_x + {}^1/_2\varkappa\eta_{xxx} + {}^1/_2\gamma\eta_{xxxx} = 0 \\ &\varkappa = {}^1/_3\mu + \delta - \beta = h(\sigma_0 - \sigma_{xx})[\rho_w g\lambda^2]^{-1} \end{aligned} \tag{1.9}$$

$$\sigma_0 = gH\,[\rho_w H/(3h) + \rho_i] \tag{1.10}$$

Let $h = 3$ m, then $H = 5$ m and we have $\sigma_0 \approx {}^2/_3 \cdot 10^5$ Nm^{-2}, while **when** $H = 10$ m, we have $\sigma \approx 2\cdot 10^5$ Nm^{-2}. The above estimates and (1.10) show that σ_{xx} can be larger, as well as smaller than σ_0, and $\varkappa$ can change its sign. We shall later show that this considerably affects the properties of the solution of Eq.(1.2). Note that when $H \gg 10$ m, the inequality $\sigma_0 \gg 10^5$ Nm^{-2} holds /6/. Therefore in the case of long waves in a liquid much deeper than 10m, we can assume that $\sigma_{xx} \ll \sigma_0$ and $\varkappa > 0$. The characteristic horizontal scale of the waves for which the condition $\mu \ll 1$ holds and the parameter γ becomes smaller, increases as the depth increases. Therefore we can neglect the elasticity of the ice when considering long waves at depths greater than tens of metres. The propagation of such waves will be described by the Korteweg-de Vries (KdV) equation.

2. Let us consider the stationary solutions of (1.9)

$$\eta(x, t) = -\zeta(\xi),\quad \xi = x - ct \tag{2.1}$$

Substituting (2.1) into (1.9), integrating twice and introducing the following new variables:

$$\zeta \equiv X, \ \zeta'^2(\xi) \equiv Y(X) \tag{2.2}$$

we obtain

$$^1/_4\gamma\,(YY'' - {}^1/_4Y'^2) + {}^1/_4\varkappa Y - {}^1/_4\varepsilon X^3 + {}^1/_2(1-c)X^2 + AX + B = 0, \ A = \text{const}, \ B = \text{const} \tag{2.3}$$

We shall seek the solution of (2.3) in the form

$$Y = \alpha_1 X^{5/2} + \alpha_2 X^2 + \alpha_3 X^{3/2} + \alpha_4 X + \alpha_5, \ X > 0, \ Y > 0 \tag{2.4}$$

Substituting (2.4) into (2.3), we obtain the exact solutions in the form

$$Y_{1,2} = \pm 2K_0 X^{5/2} - K_1 X^2 - 52B/(5\varkappa) \tag{2.5}$$

$$c \equiv c_0 = 1 - \frac{18}{169}\frac{\varkappa^2}{\gamma}, \quad A = 0, \quad K_0 = \sqrt{\frac{4\varepsilon}{35\gamma}}, \quad K_1 = \frac{4\varkappa}{\gamma}$$

$$Y_{3,4} = \pm 2K_0 X^{5/2} - K_1 X^2 \pm \frac{c - c_0}{21\gamma\sqrt{K_0}} X^{3/2} + K_2 X, \tag{2.6}$$

$$K_2 = \frac{7\varkappa(c - c_0)}{624\gamma\varepsilon}. \quad B = \frac{1}{8\gamma}K_2^2$$

Having found $Y(X)$, we can obtain the stationary **solutions** (2.1). From (2.2) we have

$$\zeta'^2 = Y_i(\zeta), \ i = 1, 2, 3, 4 \tag{2.7}$$

Eq.(2.7) has solutions of the wave type if the equation $Y_i(\tau) = 0$ $(\tau^2 = X)$ has at least three real roots $\tau_1 > \tau_2 > \tau_3$. Let $Y_i(\tau) > 0$ when $\tau_2 < \tau < \tau_1$. Then ζ will vary between τ_1^2 and τ_2^2.

Let $Y_i(\varepsilon) > 0$ when $\tau_3 < \tau < \tau_2$. Then ζ will vary between τ_2^2 and τ_3^2. When $i = 3,4$ the solution (2.7) can be expressed in terms of the elliptical Jacobi functions:

$$\zeta_{3,4} = \left[\tau_2 + (\tau_{3,1} - \tau_2)\,\mathrm{cn}^2\left(\xi\sqrt{\frac{2}{3}\frac{\tau_1 - \tau_3}{K_3}}, s_{3,4}\right)\right]^2 \tag{2.8}$$

$$s_3^2 = \frac{\tau_2 - \tau_3}{\tau_1 - \tau_3}, \quad s_4^2 = \frac{\tau_1 - \tau_2}{\tau_1 - \tau_3}$$

The period $\zeta(\xi)$ is found from the formula

$$P = 3\sqrt{\frac{K_0}{\tau_1 - \tau_0}}\,K(s)$$

where $K(s)$ is the complete elliptic integral of the first kind.

In order for Eq.(2.7) to contain solutions of the unified-wave type, it is necessary that conditions $B = 0, \varkappa < 0$ hold for $i = 1,2$. When $i = 3$, the condition $\tau_1 = \tau_2 \neq \tau_3$ must hold, while for $i = 4$ we must have the condition $\tau_3 = \tau_2 \neq \tau_1$.

Let us write $Y_{3,4}(\tau)$ in the form

$$Y_3(\tau) = 2K_0(\tau - \tau_1)^2(\tau - \tau_3) \tag{2.9}$$
$$Y_4(\tau) = 2K_0(\tau - \tau_3)^2(\tau_1 - \tau)$$

Substituting (2.9) into (2.6) we find, that τ_1 and τ_3 satisfy the quadratic equations whose discriminant is less than zero. Therefore, the unified-wave type equations are possible only when the conditions $c = c_0, B = 0, \varkappa < 0$ hold. Substituting (2.5) and (2.6) into (2.7) and taking this condition into account we obtain, after integrating,

$$\zeta = \frac{35}{169}\frac{\varkappa^2}{\gamma\varepsilon}\,\mathrm{sech}^4\left(\sqrt{-\frac{\varkappa}{52\gamma}}\,\xi\right) \tag{2.10}$$

Note that the soliton (2.10) exists only when $\varkappa < 0$. Unlike the solitons of the KdV equation, its amplitude and velocity are strictly defined. Periodic solutions (2.8) exist for any sign of $\varkappa$.

3. Let us consider the zeroth approximation to Eq.(1.9) in ε in a coordinate system moving with unit velocity. Substituting $\eta = a\exp i(kx - \omega t)$, into this equation, we obtain the dispersion relation

$$\omega = {}^1/_2\varkappa k^3 - {}^1/_2\gamma k^5 \tag{3.1}$$

which, for $\varkappa > 0$, does not contradict the condition for the wave frequencies to be synchronized /3/

$$\omega(k_1 + k_2) = \omega(k_1) + \omega(k_2) \tag{3.2}$$

When $|k_1| < \sqrt{4\lambda}$, $\lambda \equiv {}^1/_5\varkappa/\gamma$, we can find for every k_1, the value

$$k_2 = -{}^1/_2k_1 + {}^1/_2\sqrt{3}\sqrt{4\lambda - k_1^2}$$

for which condition (3.2) will be satisfied identically. The relation between k_1 and k_2 can be found graphically. The tip of the vector with coordinates (k_1, k_2) in the (k_1, k_2), plane, satisfying condition (3.2), lies on an ellipse with the centre at the origin of coordinates. The major axis of the ellipse is displaced by an angle $-\pi/4$ with respect to the k_1 axis. The magnitudes of the major and minor axes are determined from the formulas $d_1 = 2\sqrt{6\lambda}$, $d_2 = 2\sqrt{2\lambda}$.

When the magnitudes of the wave vectors of the resonantly interacting waves are known, we can construct the ellipse described above and hence determine the magnitude of the compression or tension of the ice cover.

Let us write η in the form

$$\eta = \sum_{j=1}^{3} a_j(\varepsilon t)\exp i\theta_j + \text{C. C} \tag{3.3}$$

$$\theta_1 = k_j x - \omega(k_j)t, \quad k_3 = k_1 + k_2$$

where the frequencies $\omega(k_j)$ satisfy (3.2). Substituting (3.3) into (1.9) and averaging the resulting equation over $\theta_1, \theta_2, \theta_3$, we obtain the system of equations for the complex amplitudes a_j

$$ia_1^{\cdot} = {}^3/_2k_1a_2{}^*a_3, \quad ia_2^{\cdot} = {}^3/_2k_2a_1{}^*a_3$$
$$ia_3^{\cdot} = {}^3/_2k_3a_1a_2, \quad a_j^{\cdot} \equiv da_j/d(\varepsilon t) \tag{3.4}$$

This system has been encountered earlier in other areas of physics /7/, and was first applied to waves in a deep liquid under ice in /3/. The solutions (3.4) describe the energy transfer between the harmonics $\theta_1, \theta_2, \theta_3$ over a period of time.

We can single out the following special feature of this process.

If, at $t = 0$, the energy is concentrated in the waves $\theta_1\ (\theta_2)$, i.e. if the inequality $|a_1| \gg |a_{2,3}|$ $(|a_2| \gg |a_{1,3}|)$ holds at $t = 0$, then the intensity of the wave θ_3 will be insignificant for any t, since $|a_3|$ cannot increase by more than $|a_2|k_3k_2^{-1}$ $(|a_1|k_3k_1^{-1})$. Thus the energy of a short-wave oscillation cannot increase at the expense of the long-wave oscillations.

If at $t = 0$ the energy was stored mainly within the short wave, i.e. $|a_3| \gg |a_{1,2}|$, then the pattern would become different and a_1 and a_2 may increase simultaneously at the expense of a_3.

If $k = \sqrt{\lambda}$, then a wave of length πk^{-1} will be generated with time, and solution (3.4) will become

$$a_1 = a_2 = \sqrt{2a}\,[3k\,\text{ch}\,(a\varepsilon t)]^{-1}$$
$$a_3 = 2ia\,\text{th}\,(a\varepsilon t)(3k)^{-1}, \quad a = \text{const}$$

When $t = 0$, we have $a_3 = 0$, $a_1 = a_2 = \sqrt{2}a\,(3k)^{-1}$. As $t \to \infty$, we will have $a_1 = a_2 \to 0$, $a_3 \to 2ia\,(3k)^{-1}$. Thus we have shown that the periodic solutions of (1.9) are unstable and collapse when $\varkappa > 0$.

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